Lecture 2 Outer measure L^p theory 19/05/2015.

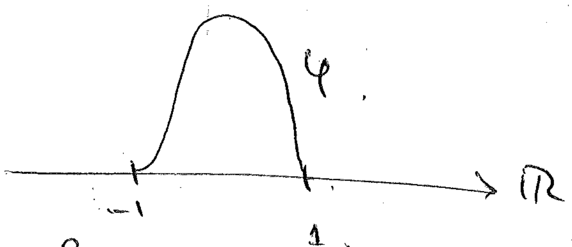
Recall:

(X,σ,S) X - metric space, $\underbrace{\sigma \text{ - premeasure on } E\in\mathbb{E}}$.
S - size $S(F,E)$. $\Downarrow$
outer measure μ on all sets.

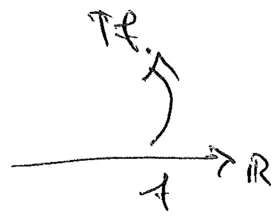
$\rightsquigarrow$ $L^p(X,\sigma,S)$. (strong L^p).
$L^{p,\infty}(X,\sigma,S)$ (weak L^p).

But quasi-normed $\to$ no exact triangle inequality.

Aim: estimate.



$$\underset{\mathbb{R}^2_+}{\underset{\wedge}{(Tf)(x,t)}} = \int \frac{1}{t}\varphi\left(\frac{x-y}{t}\right) f(y)\, dy.$$



Prop. (outer Marcinkiewicz)

Let $1\le p_1 < p_2 \le \infty$ and assume boundedness.

$$T: L^{p_1}(Y,\nu) \longrightarrow L^{p_1,\infty}(X,\sigma,S).$$
$$T: \underbrace{L^{p_2}(Y,\nu)}_{\text{classic, strong}} \longrightarrow \underbrace{L^{p_2,\infty}(X,\sigma,S)}_{\text{outer, weak}}.$$

where $|T(\lambda f)| = |\lambda T(f)|$ and
$$|T(f+g)| \le C(|Tf| + |Tg|)$$

Then, $T: L^p(X,\nu) \to \underbrace{L^p(X,\sigma,S)}_{\text{outer, strong}}$ if $p_1 < p < p_2$.

Pf. Take $f \in L^p(Y,\nu)$ s.t. $f = f_1 + f_2$, $f_i \in L^{p_i}(Y,\nu)$.
(This is true since $\nearrow$ is classical!).

Claim $\mu(S(f) > \lambda) \le \mu(S(f_1) > \lambda) + \mu(S(f_2) > \lambda)$.

~~out~~ $\operatorname{outsup}_{X\setminus F_i} S(Tf_i) \le \lambda$ and $\mu(F_i) \le \mu(S(Tf_i) > \lambda) + \varepsilon$.

$$\begin{aligned}
\operatorname{outsup}_{X\setminus F} S(Tf) &= \sup_E S(|Tf| 1_{X\setminus F}, E) \\
&\le c'\Big(\sup_E S(|Tf_1| 1_{X\setminus F}, E) + \sup_E S(|Tf_2| 1_{X\setminus F}, E)\Big) \\
&\le c'\Big[\sup_E S(|Tf_1| 1_{X\setminus F_1}, E) + \sup_E S(|Tf_2| 1_{X\setminus F_2}, E)\Big]
\end{aligned}$$

(since $1_{X\setminus F} \le 1_{X\setminus F_i}$.

$$\le 2c'\lambda.$$

So, $\mu(S(Tf) > \lambda) \le \mu(F) \le \mu(F_1) + \mu(F_2) < \dots$

$$\begin{aligned}
\|Tf\|^p_{L^p(X,\sigma,S)} &\approx \int_0^\infty \lambda^{p-1} \mu(S(Tf) > \lambda)\, d\lambda \\
&\lesssim \int_0^\infty \lambda^{p-1} \mu(S(Tf_1) > \lambda)\, d\lambda + \int_0^\infty \lambda^{p-1} \mu(S(Tf_2) > \lambda)\, d\lambda \\
&\lesssim \frac{1}{\lambda^{p_1}} \int_Y |f_1|^{p_1}\, d\nu + \frac{1}{\lambda^{p_2}} \int_Y |f_2|^{p_2}\, d\nu.
\end{aligned}$$

Take $f_1 = f \cdot 1_{|f| > \lambda}$; $f_2 = f \cdot 1_{|f| \le \lambda}$.

$$\lesssim \underbrace{\int_Y dv}_{\text{physics not.}} \, \int_0^{|f|} d\lambda \, \underbrace{(\lambda^{p-1-p_1})}_{\substack{>-1 \\ \text{if } p > p_1}} + \int_{|f|}^{\infty} d\lambda \, \underbrace{(\lambda^{p-1-p_2})}_{\substack{<-1 \\ \text{if } p < p_2}} .$$

$$\lesssim \int_Y |f|^{p_1} (|f|^{p-p_1}) \, dv + \int_Y |f|^{p_2} (|f|^{p-p_2}) \, dv .$$

$$\lesssim \int |f|^p \, dv .$$

Carleson Embedding Th^m.

Assume: ν = Carleson measure on $\mathbb{R}^2_+$, i.e.,

$\nu(E) \leq c \cdot t = c\,\sigma(E)$. $\quad (*)$

Want to prove

$$\|Tf\|_{L^p(\mathbb{R}^2_+, \nu)} \lesssim \|f\|_{L^p(\mathbb{R}, dm)} .$$

Pf (1). $\|Tf\|_{L^p(\mathbb{R}^2_+, \nu)} = \|Tf\|_{L^p(\mathbb{R}^2_+, \sigma, S_\infty)}$.

I.e., there is no one ~~measure~~ ~~actual~~ dominating measure satisfying $(*)$ with, say, $c = 1$, but there is an outer measure σ.

$$\nu(S_\infty(Tf) > \lambda) = \inf\{\nu(F) : \underbrace{\sup_E S_\infty((Tf)1_{\mathbb{R}^2_+ \setminus F}, E) \leq \lambda}_{\|Tf\|_{L^\infty(\mathbb{R}^2_+ \setminus F)} \leq \lambda}\} .$$

by looking at a large tent!

$= \nu\left(\{(x,t) : |Tf| > \lambda\}\right)$.

② $\|Tf\|_{L^p(\mathbb{R}^2_+, \nu, S_\infty)} \lesssim \|Tf\|_{L^p(\mathbb{R}^2_+, \sigma, S_\infty)}$.

$$\boxed{③ \; \|Tf\|_{L^p(\mathbb{R}^2_+, \sigma, S_\infty)} \lesssim \|f\|_{L^p(\mathbb{R})} \qquad 1 < p \le \infty.}$$

~~For this~~ Main point.

By interpolation. ~~So~~,

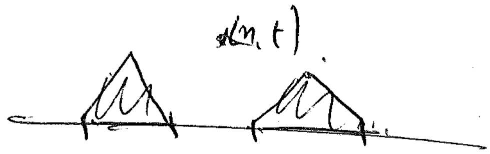
$p = \infty$: $\quad |Tf(x,t)| \le \underbrace{\left\| \frac{1}{t}\varphi\left(\frac{\cdot}{t}\right) \right\|_{L^1}}_{\le \|\varphi\|_{L^1}} \cdot \|f\|_{L^\infty}$.

$p = 1$: Need

$$\mu\left(S_\infty(Tf) > \lambda\right) \lesssim \frac{1}{\lambda}\|f\|_1$$

$\Omega := \{x : Mf > \lambda\} \subset \mathbb{R}$. $\quad$ M - H-L Max function.

$= \bigcup_i (x_i - t_i, x_i + t_i)$.



$F := \bigcup_i T(x_i, t_i)$ interval.

If $(x,t) \in F^c \Rightarrow \exists\, y \in (x-t, x+t) : (Mf)(y) \le \lambda$.

$\operatorname{outsup}_{F^c} S_\infty(Tf) = \sup_E S_\infty(Tf \cdot 1_{F^c}, E)$.

$= \sup_{F^c} |Tf|$.

$\lesssim C\, Mf(y) < \lambda$.

$$\frac{1}{\lambda}\|f\|_1 \le \mu(F) \le \sum \mu(T(x,t')) \le |\Omega| .$$ ☒

Paraproducts using S_2.

Aim: Estimate. $\quad \frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}=1.$

$$L^{p_1}(\mathbb{R})\times L^{p_2}(\mathbb{R})\times L^{p_3}(\mathbb{R}) \longrightarrow \mathbb{C}.$$

$$(f_1,f_2,f_3)\longmapsto \int_{\mathbb{R}^2_+}(T_1f_1)\cdot(T_2f_2)\cdot(T_3f_3)\,\frac{dx\,dt}{t} =: \Lambda(f_1,f_2,f_3)$$

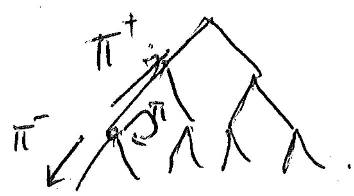
$$T_i(\cdot)=\frac{1}{t}\varphi_i\left(\frac{\cdot}{t}\right)*(\cdot).$$

Assume $\int\varphi_1=0=\int\varphi_2$.

Relation to classical SIOs:

$T=$ SIO, $\quad Tf(x)=\text{p.v.}\int k(x,y)\,f(y)\,dy.$

$$T=\Pi^{+}_{T(1)}+\Pi^{-}_{T^*(1)}+\Pi^{0}_{m}+\tilde{T}.$$



BMO
$T(1)$, $T^*(1)$.

relates to haar basis

$$\int f(\pi^+_{T(1)} g)\,dm \approx \Lambda(f, T(1), g).$$

$$\int f(\pi^-_{T^*(1)} g)\,dm \sim \Lambda(g, T^*(1), f).$$ ← no cancellation $\int - = 0$.

$$\int f(\pi^0_m g)\,dm \sim \Lambda(f, g, m).$$

Step 1:

$$|\Lambda(f_1, f_2, f_3)| \lesssim \|(T_1 f_1)\cdot(T_2 f_2)\cdot(T_3 f_3)\|_{L^1(\mathbb{R}^2_+, \sigma, S_1)}.$$

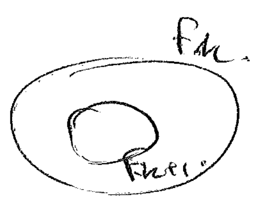
$$S_1(f, E) := \frac{1}{\sigma(E)} \int_E |f| \frac{dx\,dt}{t}.$$

Pf. Need $\left|\int g \frac{dx\,dt}{t}\right| \lesssim \|g\|_{L_1(\mathbb{R}^2_+, \sigma, S_1)}$

$$= \int_0^\infty \mu(S_1 g > \lambda)\,d\lambda.$$

$$\approx \sum_k 2^k \underbrace{\mu(S_1 g > 2^k)}_{\approx \mu(F_k)}.$$ t

where $\operatorname{outsup}_{F^c} S_1 g \le 2^k$.



$F_k \subset \bigcup_i E_k^i$; $\sum \sigma(E_k^i) \le 2\mu(F_k)$.

$$\left|\int_{\mathbb{R}^2_+} g \frac{dx\,dt}{t}\right| \le \cancel{\|g\|_{L^1(\mathbb{R}^2_+, \sigma, S_1)}}$$

$\mathbb{R}^2_+ = \bigcup_k F_k$.

$$\sum_k \int_{F_k \setminus F_{k+1}} |g| \frac{dx\,dt}{t}.$$